

Nonlinear Waves on the Surface of a Magnetohydrodynamic Fluid Column

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The method of multiple scales is used to analyse the nonlinear propagation of waves on the surface of a fluid column in the presence of a magnetic field. The evolution of the amplitude is governed by a nonlinear Schrödinger equation which gives the criterion for modulational instability. Numerical results are given in graphical form

Key words: Multiple Scales Method; Schrödinger Equation; Magnetohydrodynamic Fluid Column.

1. Introduction

The stability of a cylindrical column of fluid (the 'plasma') with an axial magnetic field has often been investigated. In particular, it has been shown that, when the plasma is confined between conducting walls, the presence of an axial magnetic field can, under suitable circumstances, stabilize the pinch. The simplest of the so-called *pinch configurations* consists of a cylindrical column of fluid (the 'plasma'), inside of which a uniform axial magnetic field is present while outside there is a similar field together with a circumferential field falling off as inverse of the radial distance from the axis of the cylinder. In the usual arrangements, the column fluid is surrounded by a concentric conducting wall. Configurations of this kind are achieved in the laboratory by sending a high current through a fluid column by means of a discharge. The axial current produces a transverse magnetic field which 'pinches' the column of fluid into a configuration which is idealized in the description in the present work.

The linear analysis of this problem was investigated earlier by Chandrasekhar [1], while the second harmonic resonance of it was investigated by Lee [2]. In this paper we study the nonlinear instability of such a fluid column. The inclusion of nonlinear terms results in amplitude modulation. In various problem of interest it has been shown that the long-time slow modulation of the wave amplitude is governed by a nonlinear Schrödinger equation.

The evolution of wave packets on the surface of a fluid column has been investigated by a number of

workers. Singh and Malik [3] investigated the weakly nonlinear breakup of a jet held together by capillary forces in the presence of an applied electric field, using the method of strained coordinates, while Trehan and Lardner [4] discussed the effect of a uniform magnetic field on the nonlinear conditions of stability of a magnetohydrodynamic jet by the method of multiple scales. The method of multiple scales was very successfully used by Hasimoto and Ono [5] to derive a single equation describing the long-time evolution of the envelope of a packet of plane finite amplitude gravity waves. In the present work we develop, by the multiple scale method the nonlinear Schrödinger equation describing the evolution of a finite amplitude wave packet on a fluid surface in the presence of an axial magnetic field with a transverse field. This method was used by Kakutani et al. [6] (see also [7, 8, 9]) to study nonlinear capillary waves on the surface of a liquid column. The method of multiple scales was also used by Khosla and Chhabra [10] to describe the nonlinear resonant interaction on a magnetohydrodynamic jet, and by Chhabra and Trehan [11] to examine weakly nonlinear progressive waves in a self gravitating fluid column in the presence of a uniform axial magnetic field.

The basic equations with the accompanying boundary conditions are given in Section 2. The first order theory and the linear dispersion relation are obtained in Section 3. In Sect. 4 we have derived second order solutions. In Sect. 5, the third order theory and the nonlinear Schrödinger equation governing the amplitude modulation are given. Finally some numerical examples are presented in graphical forms.

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2. Basic Equations

We consider axisymmetric wave motion on the surface of a fluid column whose density is ρ . Let R_0 be the radius of the fluid column and R_1 that of the encircling wall. The superscripts (1) and (2) refer to quantities inside and outside of the jet, respectively. The magnetic permeability is denoted by μ . We use cylindrical coordinates (r, φ, z) . Let $\eta(z, t)$ denote the elevation of the free surface measured from the unperturbed level $r = R_0$. The motion is assumed to be irrotational. If $\mathbf{u}$ and $\mathbf{h}$ denote velocity field and the magnetic field inside the fluid column, respectively, at any time t , then the equations holding at $r \leq R_0 + \eta$ are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p - \frac{\mu}{4\pi\rho} \mathbf{h} \times (\nabla \times \mathbf{h}), \quad (2.1)$$

$$\frac{\partial \mathbf{h}}{\partial t} = (\mathbf{h} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{h}, \quad (2.2)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2.3)$$

$$\nabla \cdot \mathbf{h}^{(1)} = 0, \quad (2.4)$$

and at $r > R_0 + \eta$ we have

$$\nabla \cdot \mathbf{h}^{(2)} = 0. \quad (2.5)$$

The unit normal $\mathbf{n}$ to the surface is given by

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = -\eta_z (\eta_z^2 + 1)^{-\frac{1}{2}} \mathbf{e}_z + (\eta_z^2 + 1)^{-\frac{1}{2}} \mathbf{e}_r, \quad (2.6)$$

where $F = 0$ is the equation of the surface of the fluid column. The condition that the interface is moving with the fluid leads to

$$\frac{\partial \eta}{\partial t} - u_r = -u_z \frac{\partial \eta}{\partial z} \quad \text{at } r = R_0 + \eta. \quad (2.7)$$

The normal component of the magnetic field is continuous at the deformed surface of the fluid column, so that

$$\mathbf{n} \cdot [\mathbf{h}] = 0 \quad \text{at } r = R_0 + \eta, \quad (2.8)$$

where $[\mathbf{h}]$ represents the jump across the surface of the fluid column, i.e., $[\mathbf{h}] = \mathbf{h}^{(2)} - \mathbf{h}^{(1)}$. At the free surface, the normal stress is continuous:

$$n_\alpha [p] - n_\beta [M_{\alpha\beta}] = 0 \quad \text{at } r = R_0 + \eta, \quad (2.9)$$

where n_α is the unit normal vector given by (2.6) and p is the pressure. The force $M_{\alpha\beta}$ of magnetic origin is

$$M_{\alpha\beta} = \mu \left(h_\alpha h_\beta - \frac{1}{2} \delta_{\alpha\beta} h_\gamma h_\gamma \right), \quad (2.10)$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. Let H_1 and H_2 denote the strengths of the axial magnetic field inside and outside of the column, respectively, and let the transverse φ -field be

$$H_\varphi = H_0 R_0 / r.$$

The fact that the magnetic fields is discontinuous at $r = R_0$ means that there is a current sheet of strength

$$J_s = \frac{1}{4\pi} \{ (H_1 - H_2) \mathbf{e}_\varphi + H_0 \mathbf{e}_z \},$$

on the surface. Furthermore, in the stationary state, the continuity of the normal stress across the surface of the fluid requires that the constant pressure p_0 inside the fluid is given by

$$p_0 = \frac{\mu}{2} (H_0^2 + H_2^2 - H_1^2). \quad (2.11)$$

It will be convenient to express H_1 and H_2 in terms of H_0 . Let

$$H_1 = \beta_1 H_0 \quad \text{and} \quad H_2 = \beta_2 H_0.$$

An inequality which must hold in virtue of (2.11) is

$$1 + \beta_2^2 \geq \beta_1^2.$$

All quantities are normalised with respect to the characteristic length R_0 , the radius of the undisturbed jet, and $1/\sqrt{\rho R_0}$. The magnetic fields are expressed as

$$\frac{\mathbf{h}^{(1)}}{H_0} = \mathbf{h},$$

$$\frac{\mathbf{h}^{(2)}}{H_0} = -\nabla \psi^{(2)} + \frac{1}{r} \mathbf{e}_\varphi.$$

Therefore,

$$\nabla^2 \psi^{(2)} = 0, \quad 1 + \eta \leq r < R_1/R_0. \quad (2.12)$$

To investigate the modulation of a weakly nonlinear wave with narrow band width spectrum, we employ the method of multiple scales by introducing the variables

$$z_n = \varepsilon^n z \quad \text{and} \quad t_n = \varepsilon^n t \quad (n = 0, 1, 2, 3),$$

and letting

$$\eta(x, t) = \sum_{n=1}^3 \varepsilon^n \eta_n(z_0, z_1, z_2; t_0, t_1, t_2) + O(\varepsilon^4), \quad (2.13)$$

$$\mathbf{u}(r, z, t) = \sum_{n=1}^3 \varepsilon^n \mathbf{u}_n(r; z_0, z_1, z_2; t_0, t_1, t_2) + O(\varepsilon^4), \quad (2.14)$$

$$p(r, z, t) = \sum_{n=1}^3 \varepsilon^n p_n(r; z_0, z_1, z_2; t_0, t_1, t_2) + O(\varepsilon^4), \quad (2.15)$$

$$\mathbf{h}(r, z, t) = \beta_1 + \sum_{n=1}^3 \varepsilon^n \mathbf{h}_n(r; z_0, z_1, z_2; t_0, t_1, t_2) + O(\varepsilon^4), \quad (2.16)$$

$$\psi^{(2)}(r, z, t) = \sum_{n=0}^3 \varepsilon^n \psi_n^{(2)}(r; z_0, z_1, z_2; t_0, t_1, t_2) + O(\varepsilon^4), \quad (2.17)$$

where the small parameter ε characterizes the steepness ratio of the wave. The short scale z_0 and fast scale t_0 denote, respectively, the wave length and the frequency of the wave. Here, t_1 and t_2 represent the slow temporal scales of the phase and amplitude, respectively, whereas the long scales z_1, z_2 stand for the spatial modulations of the phase and the amplitude. The expansions (2.13) to (2.17) are assumed to be uniformly valid for $-\infty < z < \infty$ and $0 < t < \infty$. The quantities appearing in (2.1) to (2.3) and the boundary conditions (2.7) to (2.9) can now be expressed in Maclaurin Series expansions around $r = 1$. Then, we use (2.13) to (2.17) and equate the coefficients of equal powers in ε to obtain the linear and successive nonlinear partial differential equations of various orders (see Appendix). The hierarchy of equations for each order can be derived with the knowledge of the solutions for the previous order.

3. Linear Theory

Substituting the expansions given by (2.13) to (2.17) into the field equations (2.1) to (2.3) and boundary conditions (2.7) to (2.9), and equating terms of equal powers of ε on both sides of the equation, we obtain the following set of equations and boundary conditions to order ε . The zeroth order solution yields

$$\psi_0^{(2)} = -\beta_2 z_0. \quad (3.1)$$

The velocity field and the magnetic field are derived from potential fields ϕ_1 and $\psi_1^{(1)}$ so that $\mathbf{u}_1 = \nabla_0 \phi_1$, and $\mathbf{h}_1 = -\nabla_0 \psi_1^{(1)}$, where $\nabla_0 = \mathbf{e}_r \partial/\partial r + \mathbf{e}_z \partial/\partial z_0$, and so we take

$$\psi_0^{(1)} = -\beta_1 z_0.$$

Therefore, inside of the fluid column, (A.1)–(A.4) reduce to

$$\nabla_0^2 \phi_1 = 0, \quad (3.2)$$

$$\nabla_0^2 \psi_1^{(1)} = 0, \quad (3.3)$$

$$\frac{\partial \psi_1^{(1)}}{\partial t_0} + \beta_1 \frac{\partial \phi_1}{\partial z_0} = 0, \quad (3.4)$$

$$p_1 + \frac{\partial \phi_1}{\partial t_0} = 0. \quad (3.5)$$

Outside of the fluid column we solve

$$\nabla_0^2 \psi_1^{(2)} = 0.$$

And the boundary conditions (A.5)–(A.7) reduce to

$$\frac{\partial \eta_1}{\partial t_0} - \frac{\partial \phi_1}{\partial r} = 0 \quad \text{at } r = 1, \quad (3.6)$$

$$\left[\left[\frac{\partial \psi_1}{\partial r} \right] - \frac{\partial \eta_1}{\partial z_0} \left[\left[\frac{\partial \psi_0}{\partial z_0} \right] \right] = 0 \quad \text{at } r = 1, \quad (3.7)$$

$$p_1 + \Gamma \eta_1 + \Gamma \left[\left[\beta \frac{\partial \psi_1}{\partial z_0} \right] \right] = 0, \quad \text{at } r = 1, \quad (3.8)$$

where

$$\nabla_0^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z_0^2}, \quad (3.9)$$

and Γ is the magnetic parameter defined by $\Gamma = R_0 H_0^2 \mu / 4\pi$. We obtain following solutions:

$$\eta_1 = A(z_1, z_2; t_1, t_2) \exp(i\theta) + c.c., \quad (3.10)$$

$$\phi_1 = -i \frac{\omega}{k} \frac{I_0(kr)}{I_1(k)} A(z_1, z_2; t_1, t_2) \exp(i\theta) + c.c., \quad (3.11)$$

$$\psi_1^{(1)} = -i \beta_1 \frac{I_0(kr)}{I_1(k)} A(z_1, z_2; t_1, t_2) \exp(i\theta) + c.c., \quad (3.12)$$

$$\psi_1^{(2)} = i \beta_2 \frac{K_0(kr)}{K_1(k)} A(z_1, z_2; t_1, t_2) \exp(i\theta) + c.c., \quad (3.13)$$

$$p_1 = \frac{\omega^2}{k} \frac{I_0(kr)}{I_1(k)} A(z_1, z_2; t_1, t_2) \exp(i\theta) + c.c., \quad (3.14)$$

where

$$\theta = k z_0 - \omega t_0.$$

Here, k and ω stand for the wavenumber and the frequency of the centre of the wave packet, respectively.

The progressive solutions (3.10)–(3.14) lead to the dispersion relation

$$\omega^2 = -\frac{k}{I_a} \Gamma [1 - \beta_2^2 k K_a - \beta_1^2 k I_a], \quad (3.15)$$

where

$$K_a = \frac{K_0(k)}{K_1(k)}, \quad I_a = \frac{I_0(k)}{I_1(k)}. \quad (3.16)$$

4. Second Order Solutions

Since our aim is to study the amplitude modulation when $\omega^2 > 0$, we now proceed to the second order problem in $O(\varepsilon^2)$. With the use of the first order solutions given by (3.10) to (3.14), we obtain the equations for the second order problem. If we put

$$\begin{aligned} u_2 &= \nabla \phi_2 + c_{uz}(r; z_1, z_2; t_1, t_2) e_z, \\ h_2 &= -\nabla \psi_2^{(1)} + c_{hz}(r; z_1, z_2; t_1, t_2) e_z, \end{aligned}$$

where

$$\nabla \phi_2 = \nabla_0 \phi_2 + \frac{\partial \phi_1}{\partial z_1} e_z, \quad (\phi = \phi, \psi^{(1)}),$$

we obtain the equations

$$\nabla_0^2 \phi_2 = -2\omega \frac{I_0(kr)}{I_1(k)} \frac{\partial A}{\partial z_1} \exp(i\theta) + c.c., \quad (4.1)$$

$$\nabla_0^2 \psi_2^{(1)} = -2k\beta_1 \frac{I_0(kr)}{I_1(k)} \frac{\partial A}{\partial z_1} \exp(i\theta) + c.c. \quad (4.2)$$

$$\begin{aligned} \frac{\partial \psi_2^{(1)}}{\partial t_0} + \beta_1 \frac{\partial \phi_2}{\partial z_0} &= i\beta_1 \frac{I_0(kr)}{I_1(k)} \left(\frac{\omega}{k} \frac{\partial A}{\partial z_1} + \frac{\partial A}{\partial t_1} \right) \\ &\cdot \exp(i\theta) + c.c., \end{aligned} \quad (4.3)$$

$$\begin{aligned} p_2 + \frac{\partial \phi_2}{\partial t_0} &= i \frac{\omega}{k} \frac{I_0(kr)}{I_1(k)} \frac{\partial A}{\partial t_1} - \frac{\omega^2}{2I_1^2(k)} \\ &\cdot \{I_0^2(kr) (Ae^{i\theta} + \bar{A}e^{-i\theta})^2 - I_1^2(kr) \\ &\cdot (Ae^{i\theta} - \bar{A}e^{-i\theta})^2\} \\ &- \Gamma \{\beta_1 c_{hz} - c_p(z_1, z_2; t_1, t_2)\}, \\ \nabla_0^2 \psi_2^{(2)} &= 2k\beta_2 \frac{K_0(kr)}{K_1(k)} \frac{\partial A}{\partial z_1} \exp(i\theta) + c.c., \end{aligned} \quad (4.4)$$

and the boundary conditions at $r = 1$

$$\begin{aligned} -\frac{\partial \eta_2}{\partial t_0} + \frac{\partial \phi_2}{\partial r} &= \frac{\partial A}{\partial t_1} \exp(i\theta) + i\omega(2kI_a - 1) \\ &\cdot A^2 \exp(2i\theta) + c.c., \end{aligned} \quad (4.5)$$

$$\begin{aligned} \left[\frac{\partial \psi_2}{\partial r} \right] - \frac{\partial \eta_2}{\partial z_0} \left[\frac{\partial \psi_0}{\partial z_0} \right] &= (\beta_1 - \beta_2) \frac{\partial A}{\partial z_1} e^{i\theta} \\ &- ki\{2k(\beta_1 I_a + \beta_2 K_a) + \beta_2 - \beta_1\} \\ &\cdot A^2 \exp(2i\theta) + c.c., \end{aligned} \quad (4.6)$$

$$\begin{aligned} p_2 + \Gamma \eta_2 + \Gamma \left[\beta \frac{\partial \psi_2}{\partial z_0} \right] &= -i\Gamma(\beta_2^2 K_a + \beta_1^2 I_a) \\ &\cdot \frac{\partial A}{\partial z_1} e^{i\theta} + \frac{1}{2} [-2\omega^2 + \Gamma\{\beta_2^2(K_a^2 - 3) \\ &- \beta_1^2(I_a^2 - 3)\}k^2 + 3] A^2 e^{2i\theta} \\ &+ \{\Gamma\lambda - \omega^2(I_a^2 + 1)\} A\bar{A} + c.c. - \Gamma\beta_1 c_{hz}, \end{aligned} \quad (4.7)$$

where

$$\begin{aligned} \lambda &= [\{\beta_2^2(K_a^2 - 1) - \beta_1^2(I_a^2 - 1)\}k^2 + 3] \\ &- \frac{\omega^2}{\Gamma}(1 - I_a^2). \end{aligned} \quad (4.8)$$

The non secularity conditions for the existence of a uniformly valid solution are

$$\frac{\partial A}{\partial t_1} + V_g \frac{\partial A}{\partial z_1} = 0 \quad (4.9)$$

and its complex conjugate relation. The group velocity of the wave is given by

$$\begin{aligned} V_g = \frac{d\omega}{dk} &= \frac{1}{2\omega I_a} \left\{ \omega^2 I_a \left(I_a - \frac{1}{I_a} + \frac{2}{k^2} \right) \right. \\ &\left. + \Gamma [2 + \beta_2^2 k^2 (K_a^2 - 1) - \beta_1^2 k^2 (I_a^2 - 1)] \right\}. \end{aligned} \quad (4.10)$$

The equation (4.10) shows that in the second order theory, the amplitude A is constant in a frame of reference moving with the group velocity V_g of the waves.

Equations (4.1) to (4.7) furnish the second order solutions

$$\begin{aligned} \eta_2 &= -i \left(I_a \frac{\partial A}{\partial z_1} + \frac{1}{\omega} \frac{\partial A}{\partial t_1} \right) e^{i\theta} \\ &+ \left(kI_a - \frac{1}{2} - q_1 \right) A^2 \exp(2i\theta) \\ &+ c.c. + \lambda |A|^2 - c_p(z_1, z_2; t_1, t_2), \end{aligned} \quad (4.11)$$

$$\begin{aligned} \phi_2 &= -\omega \frac{r I_1(kr)}{k I_1(k)} \frac{\partial A}{\partial z_1} e^{i\theta} \\ &+ i\omega q_1 \frac{I_0(2kr)}{k I_1(2k)} A^2 e^{2i\theta} + c.c., \end{aligned} \quad (4.12)$$

$$\begin{aligned}\psi_2^{(1)} &= \frac{\beta_1}{k I_1(k)} \\ &\cdot \left[\{-kr I_1(kr) - I_0(kr)\} \frac{\partial A}{\partial z_1} - I_0(kr) \frac{k}{\omega} \frac{\partial A}{\partial t_1} \right] e^{i\theta} \\ &+ i \beta_1 q_1 \frac{I_0(2kr)}{I_1(2k)} A^2 e^{2i\theta} + c.c.,\end{aligned}\quad (4.13)$$

$$\begin{aligned}\psi_2^{(2)} &= \frac{\beta_2}{k K_1(k)} \left[-kr K_1(kr) \frac{\partial A}{\partial z_1} + K_0(kr) \right. \\ &\cdot \left. \left\{ (1+k K_a + k I_a) \frac{\partial A}{\partial z_1} + \frac{k}{\omega} \frac{\partial A}{\partial t_1} \right\} \right] e^{i\theta} \\ &+ i \beta_2 q_2 \frac{K_0(2kr)}{K_1(2k)} A^2 e^{2i\theta} + c.c.,\end{aligned}\quad (4.14)$$

$$\begin{aligned}p_2 &= i \frac{\omega}{k I_1(k)} \left\{ -\omega r I_1(kr) \frac{\partial A}{\partial z_1} + I_0(kr) \frac{\partial A}{\partial t_1} \right\} e^{i\theta} \\ &- \frac{\omega^2}{2 I_1^2(k)} \{ I_0^2(kr) - I_1^2(kr) \} A^2 e^{2i\theta} \\ &+ 2 \{ I_0^2(kr) + I_1^2(kr) \} A \bar{A} - 2 \omega^2 q_1 \frac{I_0(2kr)}{k I_1(2k)} \\ &\cdot A^2 e^{2i\theta} + c.c. - \Gamma (\beta_1 c_{hz} - c_p),\end{aligned}\quad (4.14a)$$

where

$$q_1 = \frac{k}{D(2\omega, 2k)} \{-\Lambda + \Gamma 4k^2 \beta_2^2 (K_a + I_a) K_b\},\quad (4.15)$$

$$\begin{aligned}q_2 &= -\frac{k}{D(2\omega, 2k)} [-\Lambda + \{4I_b(-\Gamma k^2 \beta_1^2 + \omega^2) \\ &+ 2\Gamma k\} (K_a + I_a)],\end{aligned}\quad (4.16)$$

$$\begin{aligned}D(2\omega, 2k) &= \\ &-2k \left\{ I_b \frac{2\omega^2}{k} + \Gamma (1 - 2k \beta_2^2 K_b - 2k \beta_1^2 I_b) \right\},\end{aligned}\quad (4.17)$$

$$\begin{aligned}\Lambda &= \omega^2 (3 - I_a^2) - \Gamma [\{\beta_2^2 (K_a^2 - 3) \\ &- \beta_1^2 (I_a^2 - 3)\} k^2 - 2k I_a + 4],\end{aligned}\quad (4.18)$$

$$K_b = \frac{K_0(2k)}{K_1(2k)}, \quad I_b = \frac{I_0(2k)}{I_1(2k)}.\quad (4.19)$$

We have assumed that $D(2\omega, 2k) \neq 0$. The case when $D(2\omega, 2k) = 0$ corresponds to the case of the second harmonic resonance.

5. Third Order Solutions

We now proceed to the third order problem in $O(\varepsilon^2)$. The third-order problem becomes

$$\begin{aligned}\frac{\partial u_{3r}}{\partial t_0} + \frac{\partial p'_3}{\partial r} - \Gamma \beta_1 \left[\frac{\partial h_{3r}}{\partial z_0} - \frac{\partial h_{3z}}{\partial r} \right] &= \frac{i}{I_1(k)} \left[i \omega I_1(kr) \frac{\partial A}{\partial t_2} + \omega r I_0(kr) \frac{\partial^2 A}{\partial t_1 \partial z_1} \right. \\ &+ \Gamma \beta_1^2 \left\{ [kr I_0(kr) + I_1(kr)] \frac{\partial^2 A}{\partial z_1^2} + I_1(kr) \left(\frac{k}{\omega} \frac{\partial^2 A}{\partial t_1 \partial z_1} + ik \frac{\partial A}{\partial z_2} \right) \right\} \\ &\left. - \left\{ k \omega c_{uz} I_1(kr) - k \beta_1 \Gamma I_0(kr) \frac{\partial c_{hz}}{\partial r} \right\} A \right] e^{i\theta} + c.c.,\end{aligned}\quad (5.1)$$

$$\begin{aligned}\frac{\partial u_{3z}}{\partial t_0} + \frac{\partial p'_3}{\partial z_0} &= -\frac{\omega}{I_1(k)} \left[I_0(kr) \left(\frac{\partial A}{\partial t_2} + \frac{\omega}{k} \frac{\partial A}{\partial z_2} \right) - ir I_1(kr) \left(\frac{\partial^2 A}{\partial t_1 \partial z_1} + \frac{\omega}{k} \frac{\partial^2 A}{\partial z_1^2} \right) \right. \\ &\left. - \frac{i}{\omega} \left\{ -k \omega c_{uz} I_0(kr) + I_1(kr) \frac{\partial}{\partial r} (\omega c_{uz} + k \beta_1 \Gamma c_{hz}) \right\} A \right] e^{i\theta} + c.c. + M_2,\end{aligned}\quad (5.2)$$

$$\begin{aligned}\frac{\partial h_{3r}}{\partial t_0} - \beta_1 \frac{\partial u_{3r}}{\partial t_0} &= -\frac{\beta_1}{I_1(k)} \left[\omega r I_0(kr) \frac{\partial^2 A}{\partial z_1^2} + \{kr I_0(kr) + I_1(kr)\} \frac{\partial^2 A}{\partial t_1 \partial z_1} \right. \\ &\left. + I_1(kr) \left(\frac{k}{\omega} \frac{\partial^2 A}{\partial t_1^2} + ik \frac{\partial A}{\partial t_2} + i \omega \frac{\partial A}{\partial z_2} \right) - k I_1(kr) c_z A \right] e^{i\theta} + c.c.,\end{aligned}\quad (5.3)$$

$$\begin{aligned} \frac{\partial h_{3z}}{\partial t_0} - \beta_1 \frac{\partial u_{3z}}{\partial z_0} = & -\frac{\beta_1}{I_1(k)} \left[-k I_0(kr) \frac{\partial A}{\partial t_2} + i \{kr I_1(kr) + 2 I_0(kr)\} \frac{\partial^2 A}{\partial t_1 \partial z_1} \right. \\ & + I_0(kr) \left(i \frac{k}{\omega} \frac{\partial^2 A}{\partial t_1^2} - \omega \frac{\partial A}{\partial z_2} \right) + i \frac{\omega}{k} \{kr I_1(kr) + I_0(kr)\} \frac{\partial^2 A}{\partial z_1^2} \\ & \left. - i \left\{ k I_0(kr) c_z + I_1(kr) \frac{\partial c_z}{\partial r} \right\} A \right] e^{i\theta} + c.c. + M_4, \end{aligned} \quad (5.4)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r u_{3r}) + \frac{\partial u_{3z}}{\partial z_0} = \frac{\omega}{k I_1(k)} \left[i \{I_0(kr) + kr I_1(kr)\} \frac{\partial^2 A}{\partial z_1^2} - k I_0(kr) \frac{\partial A}{\partial z_2} \right] e^{i\theta} + c.c. + M_5, \quad (5.5)$$

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} (r h_{3r}) + \frac{\partial h_{3z}}{\partial z_0} = & -\frac{\beta_1}{I_1(k)} \left[i \{2 I_0(kr) + kr I_1(kr)\} \frac{\partial^2 A}{\partial z_1^2} - k I_0(kr) \left(\frac{\partial A}{\partial z_2} - \frac{i}{\omega} \frac{\partial^2 A}{\partial z_1 \partial t_1} \right) \right] \\ & \cdot \exp(i\theta) + c.c. + M_6, \end{aligned} \quad (5.6)$$

$$\begin{aligned} \nabla_0^2 \psi_3^{(2)} = & \frac{\beta_2}{K_1(k)} \left[i \{-K_0(kr) (3 + 2k K_a + 2k I_a) + 2kr K_1(kr)\} \frac{\partial^2 A}{\partial z_1^2} + 2k K_0(kr) \left(\frac{\partial A}{\partial z_2} - \frac{i}{\omega} \frac{\partial^2 A}{\partial z_1 \partial t_1} \right) \right] \\ & \cdot \exp(i\theta) + c.c., \end{aligned} \quad (5.7)$$

where

$$\begin{aligned} p'_3 = p_3 - & \frac{2\omega^2 q_1}{I_1(2k) I_1(k)} \{I_1(2kr) I_1(kr) + I_0(2kr) I_0(kr)\} A^2 \bar{A} e^{i\theta} \\ & + \frac{2i\omega^2}{k I_1^2(k)} \left\{ kr I_0(kr) I_1(kr) + \frac{1}{2} I_0^2(kr) \right\} \left(\frac{\partial \bar{A}}{\partial z_1} A - \frac{\partial A}{\partial z_1} \bar{A} \right), \end{aligned} \quad (5.8)$$

$$M_2 = \Gamma \left(\beta_1 \frac{\partial c_{hz}}{\partial z_1} - \frac{\partial c_p}{\partial z_1} \right) - \frac{\partial c_{uz}}{\partial t_1}, \quad (5.9)$$

$$M_4 = -\frac{\partial c_{hz}}{\partial t_1} + \beta_1 \frac{\partial c_{uz}}{\partial z_1} + 2\omega \left(1 - \frac{k}{\omega} V_g \right) \frac{k \beta_1}{I_1^2(k)} \{I_1^2(kr) + I_0^2(kr)\} \frac{\partial}{\partial z} A \bar{A}, \quad (5.10)$$

$$c_z = k c_{uz} + \frac{\omega}{\beta_1} c_{hz}, \quad M_5 = -\frac{\partial c_{uz}}{\partial z_1}, \quad M_6 = -\frac{\partial c_{hz}}{\partial z_1}. \quad (5.11a, b, c)$$

We seek a solution of these equations in the form

$$\begin{aligned} u_{3r} = & a_{ur} e^{i\theta} + b_{ur} + \dots, \quad u_{3z} = a_{uz} e^{i\theta} + b_{uz} + \dots, \\ h_{3r} = & a_{hr} e^{i\theta} + b_{hr} + \dots, \quad h_{3z} = a_{hz} e^{i\theta} + b_{hz} + \dots, \\ p'_3 = & a_{p'} e^{i\theta} + b_{p'} + \dots; \end{aligned} \quad (5.12)$$

where ... indicates non-secular producing terms. The term independent of θ leads to the equations

$$M_2 = 0, \quad M_4 = 0, \quad (5.13a, b)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r b_{ur}) = M_5, \quad \frac{1}{r} \frac{\partial}{\partial r} (r b_{hr}) = M_6, \quad (5.14a, b)$$

from which c_{uz} and c_{hz} can be determined. If we assume that these two quantities depend on z_1 and t_1 only through the combination of $z_1 - V_g t_1$, we obtain

$$c_{uz} = \frac{1}{\Delta} \left[\Gamma c_p V_g + \Gamma \frac{2\omega \beta_1^2 k}{I_1^2(k)} \left(1 - \frac{k}{\omega} V_g \right) \{I_1^2(kr) + I_0^2(kr)\} A \bar{A} \right], \quad (5.15)$$

$$c_{hz} = \frac{1}{\Delta} \left[-\Gamma \beta_1 c_p - \frac{2\omega \beta_1 k V_g}{I_1^2(k)} \left(1 - \frac{k}{\omega} V_g \right) \{I_1^2(kr) + I_0^2(kr)\} A \bar{A} \right], \quad (5.16)$$

where $\Delta = V_g^2 - \Gamma \beta_1^2$.

From (5.14) we can solve for b_{ur} and b_{hr} . Using (5.11b, c) and (5.15–16), we obtain

$$b_{ur} = -\frac{V_g \Gamma}{2\Delta} \frac{\partial c_p}{\partial z_1} r - \frac{2\omega \beta_1^2 \Gamma}{\Delta I_1^2(k)} \left(1 - \frac{k}{\omega} V_g\right) \frac{\partial A \bar{A}}{\partial z_1} I_1(kr) I_0(kr),$$

$$b_{hr} = \frac{\beta_1 \Gamma}{2\Delta} \frac{\partial c_p}{\partial z_1} r + \frac{2\omega \beta_1 V_g}{\Delta I_1^2(k)} \left(1 - \frac{k}{\omega} V_g\right) \frac{\partial A \bar{A}}{\partial z_1} I_1(kr) I_0(kr).$$

If we substitute (5.12) into third order equations (5.1–4), we obtain differential equations for a_{ur} , a_{uz} etc. which lead to an equation for a_{uz} by eliminating other variables. The solution a_{uz} is found to be

$$a_{uz} = -\frac{\omega}{2k I_1(k)} \left[\{r^2 k I_0(kr) + r I_1(kr)\} \frac{\partial^2 A}{\partial z_1^2} + i 2kr I_1(kr) \frac{\partial A}{\partial z_2} \right] + F_1(kr) + a_1 I_0(kr), \quad (5.17)$$

where

$$F_1(kr) = I_0(kr) \{I_1^2(kr) + I_0^2(kr) + V(kr)\} c_1 A^2 \bar{A} - \frac{1}{I_1(k)} \left\{ I_0(kr) k c_{uz} + I_1(kr) \frac{\partial c_{uz}}{\partial r} \right\} A,$$

and $V(kr)$ is defined by the equation

$$\frac{dV}{dr} = \frac{L(kr)}{r I_0^2(kr)} - k \{I_0^2(kr) + I_1^2(kr)\} \frac{I_1(kr)}{I_0(kr)}, \quad (5.18)$$

and

$$L(kr) = \int_0^{kr} \{I_0^2(x) + I_1^2(x)\}^2 x dx, \quad c_1 = \left(1 - \frac{k}{\omega} V_g\right)^2 \frac{4\omega^3 k^2 \beta_1^2 \Gamma}{I_1^3(k) \Delta (\omega^2 - \Gamma \beta_1^2 k^2)}. \quad (5.19a, b)$$

The constant a_1 in (5.17) is due to the homogeneous solution, and since the solution involving a_1 will vanish when it is substituted into the boundary condition at the free surface, it is set equal to zero. The remaining a 's can be expressed in terms of a_{uz} . In the following we record a 's which are needed in the subsequent analysis:

$$a'_p = \frac{\omega}{k I_1(k)} \left[-\frac{\omega}{2k} \{kr^2 I_0(kr) - r I_1(kr)\} \frac{\partial^2 A}{\partial z_1^2} - i \frac{\omega}{k} \{-kr I_1(kr) + I_0(kr)\} \frac{\partial A}{\partial z_2} \right. \\ \left. + r I_1(kr) \frac{\partial^2 A}{\partial z_1 \partial t_1} + i I_0(kr) \frac{\partial A}{\partial t_2} \right] + \frac{\omega}{k} F_1(kr) - \frac{1}{k I_1(k)} \left\{ \omega k I_0(kr) c_{uz} - I_1(kr) \frac{\partial}{\partial r} (c_{hz} \beta_1 k \Gamma + c_{uz} \omega) \right\} A, \quad (5.20)$$

$$a_{ur} = \frac{\omega}{k I_1(k)} \left[\frac{i}{2} \{r I_0(kr) + r^2 k I_1(kr)\} \frac{\partial^2 A}{\partial z_1^2} + \{-kr I_0(kr) + I_1(kr)\} \frac{\partial A}{\partial z_2} \right] + F_2(kr), \quad (5.21)$$

$$a_{hz} = \frac{\beta_1}{I_1(k)} \left[\frac{1}{2} \{(kr^2 + 2) I_0(kr) + 3r I_1(kr)\} \frac{\partial^2 A}{\partial z_1^2} + i \{kr I_1(kr) + I_0(kr)\} \frac{\partial A}{\partial z_2} \right. \\ \left. + \frac{1}{\omega} \{kr I_1(kr) + 2 I_0(kr)\} \frac{\partial^2 A}{\partial z_1 \partial t_1} + \frac{k}{\omega} I_0(kr) \left(i \frac{\partial A}{\partial t_2} + \frac{1}{\omega} \frac{\partial^2 A}{\partial t_1^2} \right) \right] \\ - \beta_1 \frac{k}{\omega} F_1(kr) - \frac{1}{\omega I_1(k)} \left\{ k I_0(kr) + I_1(kr) \frac{\partial}{\partial r} \right\} c_z \beta_1 A, \quad (5.22)$$

$$\psi_3^{(2)} = \frac{\beta_2}{k K_1(k)} \left[\frac{i}{2} \{r K_1(kr) (1 + 2k K_a + 2k I_a) - K_0(kr) kr^2\} \frac{\partial^2 A}{\partial z_1^2} \right. \\ \left. - kr K_1(kr) \left(\frac{\partial A}{\partial z_2} - \frac{1}{\omega} \frac{\partial^2 A}{\partial z_1 \partial t_1} \right) + J(z_1, z_2; t_1, t_2) K_0(kr) \right] e^{i\theta} + c.c., \quad (5.23)$$

where

$$F_2(kr) = -i c_1 \left\{ I_1(kr) V(kr) + \frac{L(kr)}{kr I_0(kr)} \right\} + i \frac{k I_1(kr)}{I_1(k)} c_{uz} A + c.c., \quad (5.24)$$

and $J(z_1, z_2; t_1, t_2)$ in (5.23) is an arbitrary solution which will be determined from the boundary conditions.

The boundary conditions at $r = 1$ are

$$-\frac{\partial \eta_3}{\partial t_0} + u_{3r} = \left[-i I_a \frac{\partial^2 A}{\partial z_1 \partial t_1} - i \omega^{-1} \frac{\partial^2 A}{\partial t_1^2} + \frac{\partial A}{\partial t_2} + i \omega Q_1 |A|^2 A + i \{ \omega q_6 (k I_a - 1) + k c_{uz}(1) \} A \right] e^{i\theta} + c.c. + M_8, \quad (5.25)$$

$$\left(\frac{\partial \psi_3^{(2)}}{\partial r} + h_{3r} \right) - \frac{\partial \eta_3}{\partial z_0} (\beta_1 - \beta_2) = \left[(\beta_1 - \beta_2) \left(-i I_a \frac{\partial^2 A}{\partial z_1^2} - \frac{i}{\omega} \frac{\partial^2 A}{\partial z_1 \partial t_1} + \frac{\partial A}{\partial z_2} \right) + i k Q_2 |A|^2 A + i k \{ c_{hz}(1) - q_6 q_4 \} A \right] e^{i\theta} + c.c., \quad (5.26)$$

$$p'_3 + \Gamma \eta_3 + \Gamma \left[\beta_2 \frac{\partial \psi_3}{\partial z_0} + \beta_1 h_{3z} \right] = \left[-\Gamma \beta_2^2 \left\{ K_a \frac{1}{\omega} \frac{\partial^2 A}{\partial t_1 \partial z_1} + \left[-1 + K_a \left(\frac{1}{k} + K_a + I_a \right) \right] \frac{\partial^2 A}{\partial z_1^2} + i K_a \frac{\partial A}{\partial z_2} \right\} + \{ Q_3 A \bar{A} + q_6 q_5 + \Gamma \beta_1 k I_a c_{hz}(1) \} A \right] e^{i\theta} + c.c., \quad (5.27)$$

where

$$Q_1 = \left[2q_1(1 - k I_b) + q_3(k I_a + 1) + \frac{1}{2}(3k^2 + 2 - k I_a) \right], \quad (5.28)$$

$$Q_2 = -\beta_2 \left[2q_2(1 + k K_b) + q_3(k K_a - 1) + \frac{1}{2}(3k^2 + 2 + k K_a) \right] - \beta_1 Q_1, \quad (5.29)$$

$$Q_3 = \omega^2 \left\{ \frac{3}{2} k I_a + 1 - q_1(2 I_a I_b - 3) \right\} - \Gamma \left[k^2 \left\{ 2 [\beta_2^2 q_2(1 - K_a K_b) + \beta_1^2 q_1(1 - I_a I_b)] + \left(q_3 - \frac{1}{2} \right) (\beta_2^2 - \beta_1^2) + \frac{5}{2} k (\beta_2^2 K_a + \beta_1^2 I_a) \right\} - 3q_3 + 6 \right], \quad (5.30)$$

$$M_8 = -\frac{\partial c_p}{\partial t_1} - \{ (1 + \lambda) V_g - \omega(I_a - 2k) \} \frac{\partial A \bar{A}}{\partial z_1}, \quad (5.31)$$

with

$$q_3 = k I_a - \frac{1}{2} - q_1,$$

$$q_4 = \beta_2(k K_a + 1) + \beta_1(k I_a - 1),$$

$$q_5 = -\omega^2 - \Gamma \{ k^2 (\beta_2^2 - \beta_1^2) - 3 \},$$

$$q_6 = \lambda |A|^2 - c_p.$$

In the above equations $c_{uz}(1)$ and $c_{hz}(1)$ are the values of c_{uz} and c_{hz} , respectively, at $r = 1$.

The boundary condition (5.25) furnishes

$$b_{ur} = M_8 \quad (5.32)$$

from which c_p is determined as

$$c_p = q A \bar{A},$$

$$q = \{ V_g (V_g^2 - \Gamma \beta_1^2 + \Gamma/2) \}^{-1} \cdot \left[-2\omega \Gamma \beta_1^2 I_a \left(1 - \frac{k}{\omega} V_g \right) + \Delta \{ (1 + \lambda) V_g - \omega(I_a - 2k) \} A \bar{A} \right].$$

Now if we put $\eta_3 = a_\eta e^{i\theta} + b_\eta + \dots$, and a_{ur} from (5.21) into (5.25), we find that

$$a_\eta = i(1 - k I_a) \frac{1}{k} \frac{\partial A}{\partial z_2} - (I_a + k) \frac{1}{2k} \frac{\partial^2 A}{\partial z_1^2} - \frac{i}{\omega} \frac{\partial A}{\partial t_2} - \frac{1}{\omega^2} \left(\omega I_a \frac{\partial^2 A}{\partial z_1 \partial t_1} + \frac{\partial^2 A}{\partial t_1^2} \right) + \frac{i}{\omega} F_2(k) + \{ q_6(k I_a - 1) + Q_1 |A|^2 \} A + \frac{k}{\omega} c_{uz}(1) A. \quad (5.33)$$

If we substitute from (5.33) into (5.26), $J(z_1, z_2; t_1, t_2)$ in (5.23) is determined as

$$\begin{aligned}
 J = & -i(3+2kK_a)(K_a+I_a)\frac{1}{2}\frac{\partial^2 A}{\partial z_1^2} \\
 & + k(K_a+I_a)\left(\frac{\partial A}{\partial z_2} - \frac{i}{\omega}\frac{\partial^2 A}{\partial z_1 \partial t_1}\right) - \frac{i}{\omega}\frac{\partial^2 A}{\partial t_1 \partial z_1} \\
 & - i\frac{k}{\omega^2}\frac{\partial^2 A}{\partial t_1^2} + \frac{k}{\omega}\frac{\partial A}{\partial t_2} \\
 & + ik\left\{q_7|A|^2 + q_6k(K_a+I_a) + \frac{k}{\omega}c_{uz}(1)\right\}A \\
 & - \frac{k}{\omega}F_2(k), \quad (5.34)
 \end{aligned}$$

where

$$\begin{aligned}
 q_7 = & 2q_1(1-kI_b) + 2q_2(1+kK_b) \\
 & + \left(q_3 - \frac{1}{2}\right)k(K_a+I_a).
 \end{aligned}$$

Substituting from (5.20), (5.22), (5.23), (5.33), and (5.34) into the boundary condition (5.27), we finally obtain the nonlinear Schrödinger equation

$$i\left(\frac{\partial A}{\partial t_2} + V_g\frac{\partial A}{\partial z_2}\right) + P\frac{\partial^2 A}{\partial z_1^2} = QA^2\bar{A},$$

where

$$P = \frac{1}{2}\frac{dV_g}{dk}, \quad (5.36)$$

$$\begin{aligned}
 Q = & \frac{k}{2\omega I_a}\left[Q_3 + \Gamma\left(1 - \frac{k}{\omega}V_g\right)^2 4\beta_1^2\omega^2\frac{L(k)}{\Delta I_1^4(k)}\right. \\
 & + \Gamma 2I_a(V_g\omega - \beta_1^2 k\Gamma)\frac{q}{\Delta} + \Gamma\beta_2^2 kK_a q_7 - \Gamma Q_1 \\
 & \left. + (\lambda - q)[\Gamma\{k\beta_2^2 K_a(K_a+I_a) - kI_a + 1\} + q_5]\right]. \quad (5.37)
 \end{aligned}$$

It is known that the modulational instability is characterized by the criterion $PQ < 0$, which yields the value of the wave number k_m at which the instability occurs. Such a criterion depends upon β_1 and β_2 . In Figs. 1 and 2 we show the stability regions in the $\beta_1 - k$ plane when β_2 is equal to 0.6 and 0.8, respectively. Here, β_1 can only take the values for $\beta_1^2 \leq 1 + \beta_2^2$. In Fig. 3, when $\beta_1 = 0.8$, the region of stability in the $\beta_2 - k$ plane is shown when β_2 varies from 0.3 to 1.3. Since P is positive when $\beta_1 > 0.7$, the stability depends on the sign of Q . We observe that there are two branches of the curve

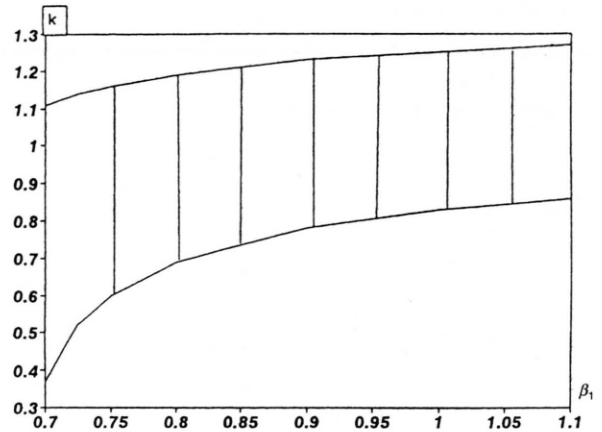


Fig. 1. Stability regions (shaded) in the case $\beta_2 = 0.6$

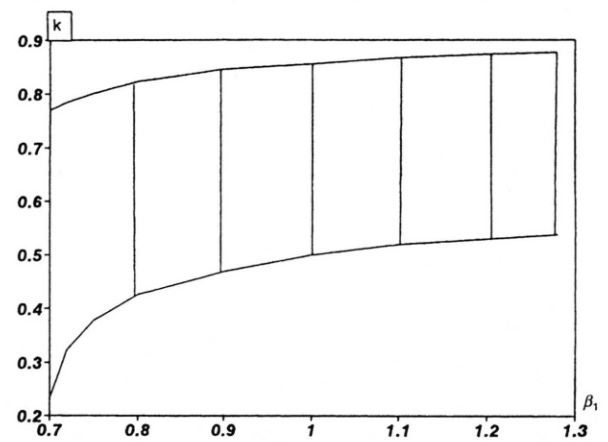


Fig. 2. Stability regions (shaded) in the case $\beta_2 = 0.8$

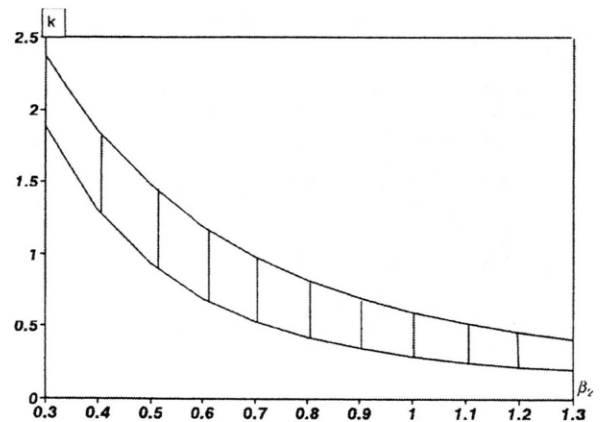


Fig. 3. Stability regions (shaded) in the case $\beta_1 = 0.8$

$Q = 0$. These branches are lying above the linear curve, forming two unstable regions and one stable region, which forms a band. The stable region, which is shaded, decreases with the increase of k .

Appendix

The first-order problem for $O(\varepsilon)$ is

$$\frac{1}{r} \frac{\partial}{\partial r} (ru_{1r}) + \frac{\partial u_{1z}}{\partial z_0} = 0, \quad (\text{A.1})$$

$$\frac{1}{r} \frac{\partial}{\partial r} (rh_{1r}) + \frac{\partial h_{1z}}{\partial z_0} = 0, \quad (\text{A.2})$$

$$\frac{\partial \mathbf{h}_1}{\partial t_0} - \beta_1 \frac{\partial \mathbf{u}_1}{\partial z_0} = 0, \quad (\text{A.3})$$

$$\frac{\partial \mathbf{u}_1}{\partial t_0} + \nabla_0 p_1 + \Gamma \beta_1 \mathbf{e}_z \times (\nabla_0 \times \mathbf{h}_1) = 0. \quad (\text{A.4})$$

Boundary conditions on $r = 1$:

$$u_{1r} - \frac{\partial \eta_1}{\partial t_0} = 0, \quad (\text{A.5})$$

$$(\beta_2 - \beta_1) \frac{\partial \eta_1}{\partial z_0} + \left[\left[\frac{\partial \psi_1}{\partial r} \right] \right] = 0 \quad (\text{A.6})$$

$$p_1 + \Gamma \eta_1 + \Gamma \left[\left[\beta \frac{\partial \psi_1}{\partial z_0} \right] \right] = 0 \quad (\text{A.7})$$

The second-order problem for $O(\varepsilon^2)$ is

$$\frac{1}{r} \frac{\partial}{\partial r} (ru_{2r}) + \frac{\partial u_{2z}}{\partial z_0} = - \frac{\partial u_{1z}}{\partial z_1}, \quad (\text{A.8})$$

$$\frac{1}{r} \frac{\partial}{\partial r} (rh_{2r}) + \frac{\partial h_{2z}}{\partial z_0} = - \frac{\partial h_{1z}}{\partial z_1}, \quad (\text{A.9})$$

$$\frac{\partial \mathbf{h}_2}{\partial t_0} - \beta_1 \frac{\partial \mathbf{u}_2}{\partial z_0} = - \frac{\partial \mathbf{h}_1}{\partial t_1} + \beta_1 \frac{\partial \mathbf{u}_1}{\partial z_1} + (\mathbf{h}_1 \cdot \nabla) \mathbf{u}_1 - (\mathbf{u}_1 \cdot \nabla) \mathbf{h}_1, \quad (\text{A.10})$$

$$\begin{aligned} \frac{\partial \mathbf{u}_2}{\partial t_0} + \nabla_0 p_2 + \Gamma \beta_1 \mathbf{e}_z \times (\nabla_0 \times \mathbf{h}_2) \\ = - \frac{\partial \mathbf{u}_1}{\partial t_1} - \frac{\partial p_1}{\partial z_1} \mathbf{e}_z - \Gamma \mathbf{h}_1 \times (\nabla_0 \times \mathbf{h}_1) \\ - (\mathbf{u}_1 \cdot \nabla) \mathbf{u}_1 - \Gamma \beta_1 \mathbf{e}_z \times \left(\mathbf{e}_z \frac{\partial}{\partial z_1} \times \mathbf{h}_1 \right). \end{aligned} \quad (\text{A.11})$$

Boundary conditions on $r = 1$:

$$u_{2r} - \frac{\partial \eta_2}{\partial t_0} = \frac{\partial \eta_1}{\partial t_1} + \eta_1 \frac{\partial u_{1r}}{\partial r} - \frac{\partial \eta_1}{\partial z_0} u_{1z}, \quad (\text{A.12})$$

$$\begin{aligned} (\beta_2 - \beta_1) \frac{\partial \eta_2}{\partial z_0} + \left[\left[\frac{\partial \psi_2}{\partial r} \right] \right] \\ = \frac{\partial \eta_1}{\partial z_0} \left[\left[\frac{\partial \psi_1}{\partial z_0} \right] \right] - \eta_1 \left[\left[\frac{\partial^2 \psi_1}{\partial r^2} \right] \right] + (\beta_1 - \beta_2) \frac{\partial \eta_1}{\partial z_1} \end{aligned} \quad (\text{A.13})$$

$$\begin{aligned} p_2 + \Gamma \eta_2 + \Gamma \left[\left[\beta \frac{\partial \psi_2}{\partial z_0} \right] \right] = - \frac{\partial p_1}{\partial r} \eta_1 \\ - \Gamma \left[\left[\frac{1}{2} \left(\frac{\partial \psi_1}{\partial r} \right)^2 + 2\beta \frac{\partial \psi_1}{\partial r} \frac{\partial \eta_1}{\partial z_0} \right. \right. \\ \left. \left. + \beta^2 \left(\frac{\partial \eta_1}{\partial z_0} \right)^2 + \beta \left(\frac{\partial^2 \psi_1}{\partial z_0 \partial r} \eta_1 + \frac{\partial \psi_1}{\partial z_1} \right) - \frac{1}{2} \left(\frac{\partial \psi_1}{\partial z_0} \right)^2 \right] \right] \\ + \Gamma \left[\frac{3}{2} \eta_1^2 - \beta_1 c_{hz} \right]. \end{aligned} \quad (\text{A.14})$$

The third-order problem for $O(\varepsilon^3)$ is

$$\begin{aligned} \frac{\partial \mathbf{u}_3}{\partial t_0} + \nabla_0 p_3 + \Gamma \beta_1 \mathbf{e}_z \times (\nabla_0 \times \mathbf{h}_3) \\ = - \frac{\partial \mathbf{u}_1}{\partial t_2} - \frac{\partial \mathbf{u}_2}{\partial t_1} - \left(\frac{\partial p_1}{\partial z_2} + \frac{\partial p_2}{\partial z_1} \right) \mathbf{e}_z \\ - \Gamma \mathbf{h}_1 \times (\nabla_0 \times \mathbf{h}_2) - \Gamma \mathbf{h}_2 \times (\nabla_0 \times \mathbf{h}_1) \\ - (\mathbf{u}_1 \cdot \nabla) \mathbf{u}_2 - (\mathbf{u}_2 \cdot \nabla) \mathbf{u}_1 - u_{1z} \frac{\partial \mathbf{u}_1}{\partial z_1} \\ - \Gamma \beta_1 \mathbf{e}_z \times \left(\mathbf{e}_z \frac{\partial}{\partial z_1} \times \mathbf{h}_2 \right) \\ - \Gamma \beta_1 \mathbf{e}_z \times \left(\mathbf{e}_z \frac{\partial}{\partial z_2} \times \mathbf{h}_1 \right) \\ - \Gamma \mathbf{h}_1 \times \left(\mathbf{e}_z \frac{\partial}{\partial z_1} \times \mathbf{h}_1 \right), \end{aligned} \quad (\text{A.15})$$

$$\begin{aligned} \frac{\partial \mathbf{h}_3}{\partial t_0} - \beta_1 \frac{\partial \mathbf{u}_3}{\partial z_0} = - \frac{\partial \mathbf{h}_1}{\partial t_2} - \frac{\partial \mathbf{h}_2}{\partial t_1} \\ + \beta_1 \left(\frac{\partial \mathbf{u}_1}{\partial z_2} + \frac{\partial \mathbf{u}_2}{\partial z_1} \right) + h_{1z} \frac{\partial \mathbf{u}_1}{\partial z_1} \\ - u_{1z} \frac{\partial \mathbf{h}_1}{\partial z_1} + (\mathbf{h}_1 \cdot \nabla) \mathbf{u}_2 + (\mathbf{h}_2 \cdot \nabla) \mathbf{u}_1 \\ - (\mathbf{u}_1 \cdot \nabla) \mathbf{h}_2 - (\mathbf{u}_2 \cdot \nabla) \mathbf{h}_1, \end{aligned} \quad (\text{A.16})$$

$$\frac{1}{r} \frac{\partial}{\partial r} (ru_{3r}) + \frac{\partial u_{3z}}{\partial z_0} = - \frac{\partial u_{1z}}{\partial z_2} - \frac{\partial u_{2z}}{\partial z_1}, \quad (\text{A.17})$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r h_{3r}) + \frac{\partial h_{3z}}{\partial z_0} = -\frac{\partial h_{1z}}{\partial z_2} - \frac{\partial h_{2z}}{\partial z_1}. \quad (\text{A.18})$$

Boundary conditions on $r = 1$:

$$\begin{aligned} u_{3r} - \frac{\partial \eta_3}{\partial t_0} &= \frac{\partial \eta_1}{\partial t_2} + \frac{\partial \eta_2}{\partial t_1} - \eta_1 \frac{\partial u_{2r}}{\partial r} - \eta_2 \frac{\partial u_{1r}}{\partial r} \\ -\frac{1}{2} \eta_1^2 \frac{\partial^2 u_{1r}}{\partial r^2} + \frac{\partial \eta_1}{\partial z_0} u_{2z} + \left(\frac{\partial \eta_1}{\partial z_1} + \frac{\partial \eta_2}{\partial z_0} \right) u_{1z}, \end{aligned} \quad (\text{A.19})$$

$$\begin{aligned} &(\beta_2 - \beta_1) \frac{\partial \eta_3}{\partial z_0} + \left[\frac{\partial \psi_3^{(2)}}{\partial r} + h_{3r} \right] \\ &= \frac{\partial \eta_1}{\partial z_0} \left(\left[\frac{\partial \psi_2}{\partial z_0} \right] + \eta_1 \left[\frac{\partial^2 \psi_1}{\partial z_0 \partial r} \right] + \left[\frac{\partial \psi_1}{\partial z_1} \right] \right) \\ &+ (\beta_1 - \beta_2) \left(\frac{\partial \eta_1}{\partial z_2} + \frac{\partial \eta_2}{\partial z_1} \right) + \frac{\partial \eta_1}{\partial z_1} \left[\frac{\partial \psi_1}{\partial z_0} \right] \\ &+ \frac{\partial \eta_2}{\partial z_0} \left[\frac{\partial \psi_1}{\partial z_0} \right] - \eta_1 \left[\frac{\partial^2 \psi_2}{\partial r^2} \right] \\ &- \frac{\eta_1^2}{2} \left[\frac{\partial^3 \psi_1}{\partial r^3} \right] - \eta_2 \left[\frac{\partial^2 \psi_1}{\partial r^2} \right] + \frac{\partial \eta_1}{\partial z_0} c_{hz}, \end{aligned} \quad (\text{A.20})$$

$$\begin{aligned} &p_3 + \Gamma \eta_3 + \Gamma \left[\beta_2 \frac{\partial \psi_3^{(2)}}{\partial z_0} + \beta_1 h_{3z} \right] = -\frac{\partial p_2}{\partial r} \eta_1 \\ &- \frac{\partial p_1}{\partial r} \eta_2 - \frac{\partial^2 p_1}{\partial r^2} \frac{\eta_1^2}{2} - \Gamma \left[\frac{\partial \psi_1}{\partial r} \left(\eta_1 \frac{\partial^2 \psi_1}{\partial r^2} + \frac{\partial \psi_2}{\partial r} \right) \right. \\ &+ 2 \left\{ \beta \left(\frac{\partial^2 \psi_1}{\partial r^2} \eta_1 + \frac{\partial \psi_2}{\partial r} \right) \frac{\partial \eta_1}{\partial z_0} \right. \\ &+ \beta \frac{\partial \psi_1}{\partial r} \left(\frac{\partial \eta_2}{\partial z_0} + \frac{\partial \eta_1}{\partial z_1} \right) - \frac{\partial \psi_1}{\partial r} \frac{\partial \psi_1}{\partial z_0} \frac{\partial \eta_1}{\partial z_0} \left. \right\} \\ &+ 2 \left\{ \beta^2 \frac{\partial \eta_1}{\partial z_0} \left(\frac{\partial \eta_2}{\partial z_0} + \frac{\partial \eta_1}{\partial z_1} \right) - \beta \frac{\partial \psi_1}{\partial z_0} \left(\frac{\partial \eta_1}{\partial z_0} \right)^2 \right\} \\ &+ \beta \left(\frac{1}{2} \frac{\partial^3 \psi_1}{\partial z_0 \partial r^2} \eta_1^2 + \frac{\partial^2 \psi_1}{\partial z_1 \partial r} \eta_1 + \frac{\partial^2 \psi_2}{\partial z_0 \partial r} \eta_1 \right. \\ &+ \frac{\partial^2 \psi_1}{\partial z_0 \partial r} \eta_2 + \frac{\partial \psi_2}{\partial z_1} + \frac{\partial \psi_1}{\partial z_2} \left. \right) \\ &- \frac{\partial \psi_1}{\partial z_0} \left(\frac{\partial \psi_2}{\partial z_0 \partial r} \eta_1 + \frac{\partial \psi_1}{\partial z_1} + \frac{\partial \psi_2}{\partial z_0} \right) \left. \right] \quad (\text{A.21}) \\ &+ \Gamma \left[3\eta_1 \eta_2 - 2\eta_1^3 - \beta_1 \frac{\partial c_{hz}}{\partial r} \eta_1 + \frac{\partial \psi_1^{(1)}}{\partial z_0} c_{hz} \right]. \end{aligned}$$

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